Module-2: Contour Integration-II

We have already considered integrands which have no poles on the real axis. Now we consider the integrands which may have poles on the real axis as well as inside the semi-circle C_R. The poles on the real axis are excluded by detouring them with semi-circles of small radii. This procedure is known as indenting at a point.

Jordan Inequality

If $0 \leq \theta \leq \pi/2$, then $\frac{2}{\pi} \leq \frac{\sin\,\theta}{\theta} \leq 1$. This inequality is known as Jordan's inequality.

Proof. It will be sufficient to show that $\frac{\sin\,\theta}{\theta}$ decreases as θ increases for $\theta \in [0, \pi/2]$. This happens if

$$\frac{d}{d\theta}\left(\frac{\sin\,\theta}{\theta}\right) \leq 0 \;\; for \;\; 0 \leq \theta \leq \frac{\pi}{2}.$$

But

$$\frac{d}{d\theta}\left(\frac{\sin\,\theta}{\theta}\right) = \frac{\theta\cos\,\theta - \sin\,\theta}{\theta^2} \leq 0 \;\; whenever \;\; \theta\cos\,\theta - \sin\,\theta \leq 0.$$

Since $[\theta\cos\,\theta - \sin\,\theta]_{\theta=0} = 0$, it is now enough to note that, on $(0, \pi/2]$, the function $\theta\cos\,\theta - \sin\,\theta$ has a non-positive derivative and so decreases as θ increases. This proves the result.

Example 1. *Evaluate* $\int_0^\infty \frac{\sin\,mx}{x}dx, \;\; (m > 0)$.

Solution. *We consider the integral*

$$\int_C \frac{e^{imz}}{z}dz = \int_C f(z)dz, \; say,$$

where C *is the contour consisting of*

(i) the real axis from r *to* R*;*

(ii) the upper half of the circle $C_R : |\, z \,| = R$*;*

(iii) the real axis from $-R$ *to* $-r$*;*

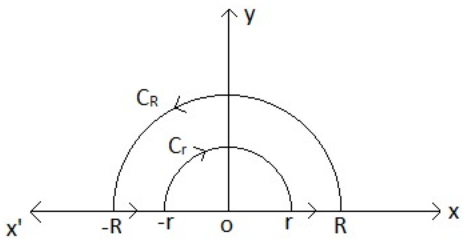

Fig. 1:

(iv) the upper half of the circle C_r : $|z| = r$ (see Fig 1.4).

Since $f(z)$ has only singularity at $z = 0$ which is outside the closed contour C, by Cauchy's fundamental theorem we get

$$0 = \int_C f(z)dz = \int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz. \quad (1)$$

On C_R we have $z = Re^{i\theta}$, $0 \le \theta \le \pi$ so that $dz = iRe^{i\theta}d\theta$. Therefore using Jordan's inequality we deduce

$$\begin{aligned}
\left| \int_{C_R} f(z)dz \right| &= \left| \int_0^\pi \frac{e^{imRe^{i\theta}}}{Re^{i\theta}} \cdot iRe^{i\theta}d\theta \right| = \left| \int_0^\pi e^{imR(\cos\theta + i\sin\theta)}d\theta \right| \\
&\le \int_0^\pi \left| e^{imR(\cos\theta + i\sin\theta)} \right| d\theta = \int_0^\pi e^{-mR\sin\theta}d\theta \\
&= 2\int_0^{\pi/2} e^{-mR\sin\theta}d\theta \le 2\int_0^{\pi/2} e^{-\frac{2mR}{\pi}\theta}d\theta \\
&= \frac{\pi}{mR}\left[e^{-\frac{2mR}{\pi}\theta}\right]_{\pi/2}^0 = \frac{\pi}{mR}(1 - e^{-mR}) \\
&\to 0 \text{ as } R \to \infty.
\end{aligned}$$

Since $z = 0$ is a simple pole of $f(z)$, $f(z)$ has a Laurent's series expansion near $z = 0$ of the form

$$f(z) = \phi(z) + \frac{a}{z}$$

where $\phi(z)$ is analytic at $z = 0$ and

$$a = Res(f;0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} e^{imz} = 1.$$

Hence

$$\int_{C_r} f(z)dz = \int_{C_r} \phi(z)dz + \int_{C_r} \frac{1}{z}dz.$$

On C_r, $z = re^{i\theta}$, $\pi \geq \theta \geq 0$ *so that* $dz = ire^{i\theta}d\theta$. *Then*

$$\begin{aligned}\int_{C_r} f(z)dz &= \int_{C_r} \phi(z)dz + \int_{\pi}^{0} id\theta \\ &= \int_{C_r} \phi(z)dz - i\pi.\end{aligned}$$

Since $\phi(z)$ *is analytic at* $z = 0$, *there exist a positive number* M *such that* $|\phi(z)| \leq M$ *in some neighbourhood of* $z = 0$. *We choose* r *so small that* C_r *lies entirely in this neighbourhood. Hence* $|\phi(z)| \leq M$ *on* C_r. *Therefore by ML-formula we have*

$$\left|\int_{C_r} \phi(z)dz\right| \leq M \cdot \pi r \to 0 \text{ as } r \to 0.$$

Thus $\lim_{r\to 0}\int_{C_r} f(z)dz = -i\pi$. *Hence proceeding to the limit as* $r \to 0$ *and* $R \to \infty$ *we obtain from (1)*

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= i\pi \\ \text{i.e.} \int_{-\infty}^{\infty} \frac{e^{imx}}{x}dx &= i\pi \\ \text{i.e.} \int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x}dx &= i\pi.\end{aligned}$$

Equating the imaginary part in both side we obtain

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x}dx = \pi$$

$$\text{i.e.} \int_{0}^{\infty} \frac{\sin mx}{x}dx = \pi/2.$$

This completes the solution.

Integration around a branch point

Example 2. *Evaluate* $\int_0^\infty \frac{x^{a-1}}{1+x}dx$ $(0 < a < 1)$.

Solution. *Let* $f(z) = \frac{z^{a-1}}{1+z}$, *where* z^{a-1} *denotes the principal branch of the multi-valued function* z^{a-1}. *We integrate* $f(z)$ *around a closed contour* C *consisting of*

(i) $L_1 : z = \rho e^{i0}$, $r \leq \rho \leq R$;

(ii) $\Gamma_R : |z| = R$;

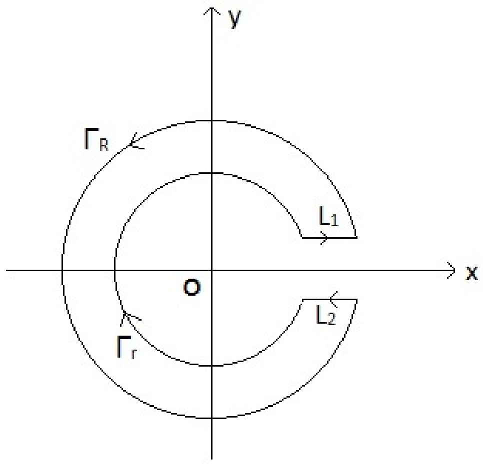

Fig. 2:

(iii) $L_2 \; : \; z \; = \rho e^{i2\pi}, \;\; R \; \geq \rho \; \geq r;$

(iv) $\Gamma_r \; : | \; z \; | = r$ *(see Fig 1.5).*

We choose L_1 *and* L_2 *sufficiently closed to the real axis so that all the poles not on the positive real axis lie within* C*. By the above choice of contour* C*, the branch point* $z \; = 0$ *of* z^{a-1} *that is of* $f(z)$ *is avoided. Now* f *has only one simple pole at* $z \; = -1$ *which lie inside* C*. Therefore by Cauchy's residue theorem we have*

$$\int_C f(z)dz \; = 2\pi i Res(f; -1). \tag{2}$$

Now

$$\begin{aligned} Res(f; -1) \;\; &= \;\; \lim_{z \to -1} (z+1)f(z) \; = \lim_{z \to -1} z^{a-1} \\ &= \;\; (-1)^{a-1} \; = e^{(a-1)\pi i} \; = -e^{a\pi i}. \end{aligned}$$

So from (2) we obtain

$$\begin{aligned} -2\pi i e^{a\pi i} \;\; &= \;\; \int_{L_1} f(z)dz + \int_{\Gamma_R} f(z)dz + \int_{L_2} f(z)dz + \int_{\Gamma_r} f(z)dz \\ &= \;\; I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3}$$

Now

$$I_1 \; = \int_r^R f(\rho)d\rho \; = \int_r^R \frac{\rho^{a-1}}{1+\rho} d\rho \; = \int_r^R \frac{x^{a-1}}{1+x} dx.$$

$$\begin{aligned} I_3 &= \int_R^r \frac{\rho^{a-1}e^{2\pi i(a-1)}}{1+\rho e^{2\pi i}} e^{2\pi i} d\rho = -\int_r^R \frac{\rho^{a-1}e^{2\pi a i}}{1+\rho} d\rho \\ &= -e^{2\pi a i}\int_r^R \frac{x^{a-1}}{1+x} dx. \end{aligned}$$

On Γ_R,

$$| f(z) | = | \frac{z^{a-1}}{1+z} | \leq \frac{| z |^{a-1}}{| z | -1} = \frac{R^{a-1}}{R-1}.$$

Applying ML-formula we see that

$$\begin{aligned} | I_2 | &= | \int_{\Gamma_R} \frac{z^{a-1}}{1+z} dz | \leq \frac{R^{a-1}}{R-1} \cdot 2\pi R \\ &= \frac{2\pi R^a}{R-1} \to 0 \; as \; R \to \infty \;\; [since \; 0 < a < 1]. \end{aligned}$$

On Γ_r,

$$| f(z) | = | \frac{z^{a-1}}{1+z} | \leq \frac{| z |^{a-1}}{1 - | z |} = \frac{r^{a-1}}{1-r}.$$

Therefore, applying ML-formula we see that

$$\begin{aligned} | I_4 | &= | \int_{\Gamma_r} \frac{z^{a-1}}{1+z} dz | \leq \frac{r^{a-1}}{1-r} \cdot 2\pi r \\ &= \frac{2\pi r^a}{1-r} \to 0 \; as \; r \to 0 \;\; [since \; 0 < a < 1]. \end{aligned}$$

So from (3) we obtain

$$(1 - e^{2\pi a i}) \int_0^\infty \frac{x^{a-1}}{1+x} dx = -2\pi i e^{\pi a i}.$$

$$\begin{aligned} i.e. \;\; \int_0^\infty \frac{x^{a-1}}{1+x} dx &= \frac{2\pi i e^{\pi a i}}{e^{2\pi a i} - 1} \\ &= \frac{2\pi i}{e^{\pi a i} - e^{-\pi a i}} = \frac{\pi}{\sin \; \pi a}. \end{aligned}$$

This completes the solution.

Example 3. *Show by the method of contour integration*

$$\int_0^\infty \sin \; x^2 dx = \frac{\sqrt{2\pi}}{4} = \int_0^\infty \cos \; x^2 dx.$$

Solution. *Let* $f(z) = e^{-z^2}$. *We integrate* f *around a closed contour* C *consisting of*

(i) the line segment $L_1 : z = x, \; 0 \leq x \leq R$;

(ii) the circular arc $C_R : z = Re^{i\theta}, \; 0 \leq \theta \leq \pi/4$;

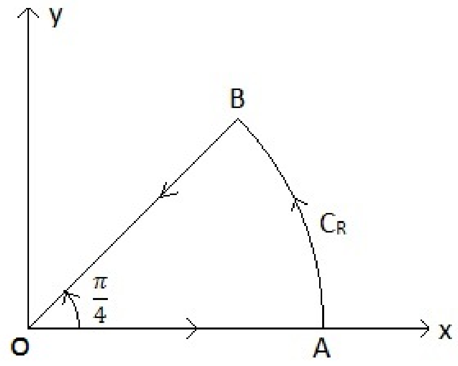

Fig. 3:

(iii) the line segment $L_2 : z = xe^{\pi i/4}, \ R \geq x \geq 0$ *(see Fig 1.6).*

Since f *is analytic within and on the closed contour* C, *by Cauchy's fundamental theorem we obtain*

$$\begin{aligned} 0 &= \int_C f(z)dz = \int_{L_1} e^{-z^2}dz + \int_{C_R} e^{-z^2}dz + \int_{L_2} e^{-z^2}dz \\ &= \int_0^R e^{-x^2}dx + \int_{C_R} e^{-z^2}dz - e^{\pi i/4}\int_0^R e^{-ix^2}dx. \end{aligned} \qquad (4)$$

Now on C_R, $z = Re^{i\theta}$, $0 \leq \theta \leq \pi/4$. *So*

$$\begin{aligned} \left| \int_{C_R} e^{-z^2}dz \right| &= \left| \int_0^{\pi/4} e^{-R^2e^{2i\theta}} \cdot iRe^{i\theta}d\theta \right| \\ &\leq R\int_0^{\pi/4} \left| e^{-R^2(\cos 2\theta + i\sin 2\theta)} \right| d\theta \\ &= R\int_0^{\pi/4} e^{-R^2\cos 2\theta}d\theta \\ &= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}d\phi, \quad \phi = \pi/2 - 2\theta. \end{aligned}$$

For $0 \leq \phi \leq \pi/2$, *we get by Jordan's inequality*

$$\frac{2\phi}{\pi} \leq \sin\ \phi$$

$$i.e.\ \ e^{-R^2\sin\ \phi} \leq e^{-R^2\frac{2\phi}{\pi}}.$$

So from above we get

$$\left| \int_{C_R} e^{-z^2} dz \right| \leq \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \frac{2\phi}{\pi}} d\phi$$
$$= \frac{\pi}{4R}\left(1 - \frac{1}{e^{R^2}}\right) \to 0 \text{ as } R \to \infty.$$

Now we consider the integral $\int_0^\infty e^{-x^2} dx$. *Putting* $x^2 = t$ *we get*

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-t} \cdot t^{\frac{1}{2}-1} dt = \frac{\Gamma(1/2)}{2} = \frac{\sqrt{\pi}}{2}.$$

Proceeding to the limit as $R \to \infty$ *we get from (4)*

$$e^{\frac{\pi i}{4}} \int_0^\infty e^{-ix^2} dx = \frac{\sqrt{\pi}}{2}$$

$$i.e. \ (\cos\ \pi/4 + i \sin\ \pi/4) \int_0^\infty (\cos\ x^2 - i \sin\ x^2) dx = \frac{\sqrt{\pi}}{2}$$

$$i.e. \ \int_0^\infty (1+i)(\cos\ x^2 - i \sin\ x^2) dx = \sqrt{\frac{\pi}{2}}.$$

Equating the real part and imaginary part we obtain

$$\int_0^\infty (\cos\ x^2 + \sin\ x^2) dx = \sqrt{\frac{\pi}{2}}$$

$$and \ \int_0^\infty (\cos\ x^2 - \sin\ x^2) dx = 0.$$

Adding and subtracting above two equalities we obtain

$$\int_0^\infty \sin\ x^2 dx = \frac{\sqrt{2\pi}}{4} = \int_0^\infty \cos\ x^2 dx.$$

This completes the solution.